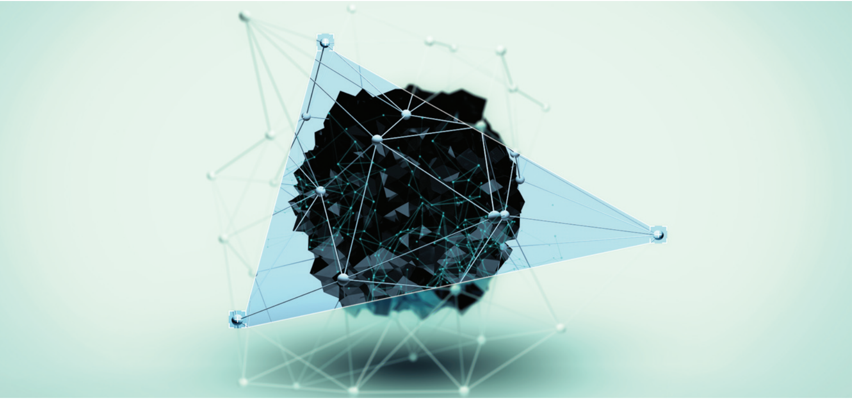
Zanimljiv dokaz kosinusova poučka



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Neka je ABC proizvoljan trokut i neka su a, b i c duljine njegovih stranica i α, β i γ njegovi kutovi. Tada vrijedi

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta \quad \text{(kosinusov poučak)}$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Dokaz: Neka je dan šiljastokutan trokut ABC i neka je $|AB| = c$, $|BC| = a$, $|CA| = b$. Ako su D, E i F dirališta trokutu ABC upisane kružnice $k_u(U, r)$, tada je $|UD| = |UE| = |UF| = r$.

Neka je $|CD| = x$ (na temelju poučka o jednakosti tangentnih dužina) slijedi da je $|CE| = |CD| = x$.

Iz slike 1 vidimo da je $|AE| = |AF| = b - x$ $|BF| = |BD| = a - x$, pa je

$$|AB| = |AF| + |FB| = c = b - x + a - x$$
$$\iff x = \frac{a+b-c}{2} \qquad (1)$$

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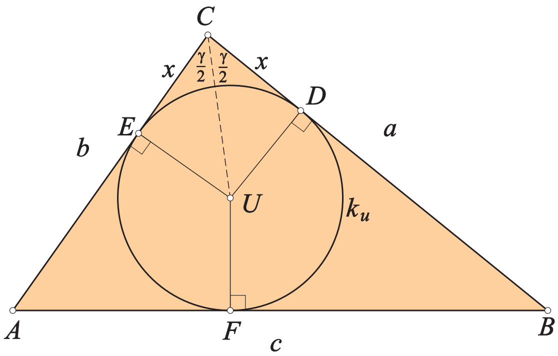


Slika 1.

U pravokutnom trokutu UDC je

$$\operatorname{tg}\frac{\gamma}{2} = \frac{r}{x} \iff x = \frac{r}{\operatorname{tg}\frac{\gamma}{2}}. \qquad (2)$$

Tako je

$$\frac{a+b-c}{2} = \frac{r}{\operatorname{tg}\frac{\gamma}{2}} \iff r = \frac{1}{2}\operatorname{tg}\frac{\gamma}{2}(a+b-c). \qquad (3)$$

Izrazimo površinu trokuta ABC na dva načina:

$$P = \frac{1}{2}ab\sin\gamma \text{ i } P = r\cdot s = r\cdot\frac{a+b+c}{2}.$$

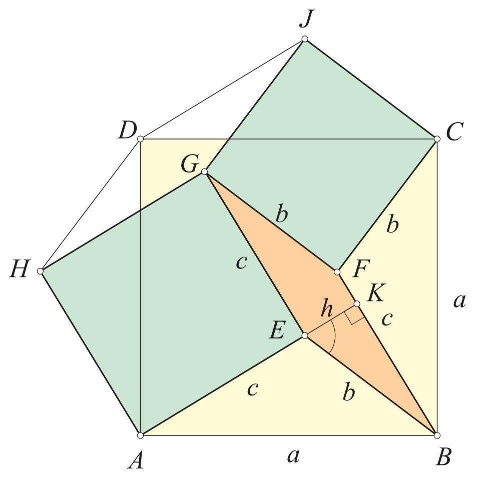
Sada imamo:

$$\frac{1}{2}ab\sin\gamma = r\cdot\frac{a+b+c}{2}$$
$$\iff r = \frac{ab\sin\gamma}{a+b+c}\ \left(\text{zbog } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$
$$= \frac{2ab\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{a+b+c},\ \text{tj. zbog (3) vrijedi}$$
$$\frac{1}{2}\operatorname{tg}\frac{\gamma}{2}(a+b-c) = \frac{2ab\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{a+b+c}$$
$$\iff \frac{1}{2}\frac{\sin\frac{\gamma}{2}}{\cos\frac{\gamma}{2}}(a+b-c) = \frac{2ab\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{a+b+c}$$
$$\iff (a+b-c)(a+b+c) = 4ab\cos^2\frac{\gamma}{2}$$
$$\left(\text{zbog } \cos^2\frac{\gamma}{2} = \frac{1+\cos\gamma}{2}\right)$$
$$\iff (a+b)^2 - c^2 = 2ab + 2ab\cos\gamma$$
$$\iff c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Ostale jednakosti ovog poučka dobivamo potpuno analogno.

Neka je sada trokut ABE tupokutan. Prikažimo geometrijski dokaz.

Uvedimo oznake kao na slici 2, $a = |AB|$, $b = |BE|$, $c = |AE|$ i neka P_{XYZ} općenito označava površinu trokuta XYZ, P_{PQRS} površinu četverokuta $PQRS$ itd.



Slika 2.

Nad stranicom $\overline{AB}$ konstruirajmo kvadrat $ABCD$, a nad stranicama $\overline{BC}$, $\overline{CD}$ i $\overline{AD}$ trokute BCF, CDJ i ADH podudarne s trokutom ABE.

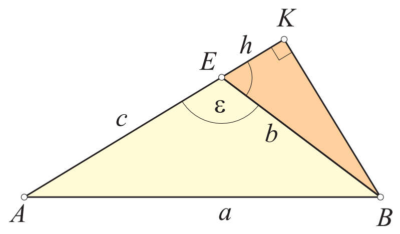
Tako je $P_{AEB} = P_{DJC}$ i $P_{BCF} = P_{ADH}$.

Zato je $P_{ABCD} = P_{AEBFCJDH}$, tj.

$$a^2 = c^2 + b^2 + 2P_{BEGF}$$
$$= c^2 + b^2 + 2\cdot|GE|\cdot h$$
$$= c^2 + b^2 + 2\cdot c\cdot h$$
$$= c^2 + b^2 + 2\cdot c\cdot b\cdot\cos(180^\circ - \varepsilon)$$
$$= c^2 + b^2 - 2bc\cos\varepsilon.$$

Pojašnjenje:

$P_{BEGF} = 2\cdot P_{BFE} = 2\cdot\frac{1}{2}\cdot c\cdot h = c\cdot h = |GE|\cdot h$, odnosno $2\cdot P_{BEGF} = 2\cdot|GE|\cdot h$ (slika 3).

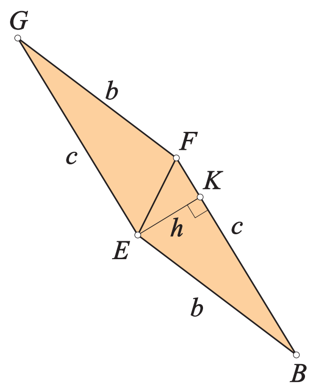


Slika 3.

U pravokutnom trokutu BKE (slika 4) je

$$\cos(180^\circ - \varepsilon) = \frac{|EK|}{|BE|} = \frac{h}{b}$$

pa zbog $\cos(180^\circ - \varepsilon) = -\cos\varepsilon$ vrijedi da je $h = -b\cos\varepsilon$. Stoga je $2\cdot c\cdot h = -2cb\cos\varepsilon$.



Slika 4.