Koliko je sin 9° i cos 9°?

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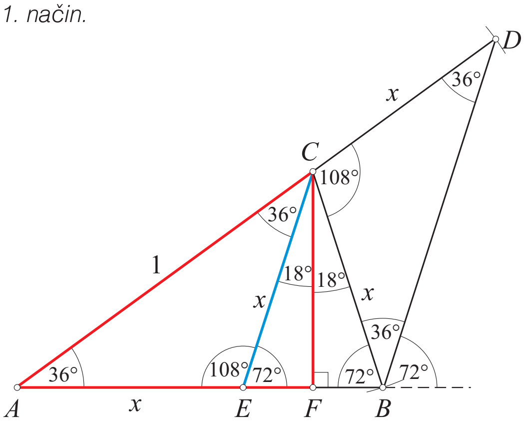
Trigonometrija je jedan od najzanimljivijih dijelova matematike, no zbog malog broja sati (3 sata tjedno u općoj gimnaziji) ne ostaje puno vremena za nadogradnju, za alternativne načine rješavanja i zadatke/probleme primjerene natjecanjima.

Meni su posebno zanimljivi zadatci u kojima se traži točno rješenje bez neposredne uporabe džepnog računala. Nedavno sam s cijenjenim profesorom Vinkom Bajrovićem rješavao i raspravljao ovaj zadatak:

Zadatak. Bez uporabe džepnog računala odredi $\sin 9^\circ$, $\cos 9^\circ$.

1. način.



$$|AB| = 1 = |AC|, \quad \overline{EC} \| \overline{BD}$$
$$\Longrightarrow |EB| : |AE| = |CD| : |AC|$$
$$\Longrightarrow (1-x) : x = x : 1$$

$\Longrightarrow$ stranica $\overline{AB}$ je točkom E podijeljena po zlatnom rezu

$$\Longrightarrow \boxed{|AE| = x = \frac{-1+\sqrt{5}}{2}.}$$

Nadalje je

$$|EB| = 1 - x = 1 - \frac{-1+\sqrt{5}}{2} = \frac{2+1-\sqrt{5}}{2}$$

$$\Longrightarrow \boxed{|EB| = x = \frac{3-\sqrt{5}}{2}.}$$

$$|EF| = \frac{1}{2}|EB| \Longrightarrow \boxed{|EF| = \frac{3-\sqrt{5}}{4}.}$$

Pitagorin poučak u trokutu AFC daje

$$\begin{aligned} |CF|^2 &= |AC|^2 - |AF|^2 = 1^2 - (|AE| + |EF|)^2 \\ &= 1 - \left(\frac{-1+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{4}\right)^2 \\ &= 1 - \left(\frac{-2+2\sqrt{5}+3-\sqrt{5}}{4}\right)^2 \\ &= 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2 \end{aligned}$$

$$|CF|^2 = \frac{16-(1+2\sqrt{5}+5)}{16} = \frac{10-2\sqrt{5}}{16}$$

$$\Longrightarrow \boxed{|CF| = \frac{1}{4}\sqrt{10-2\sqrt{5}}}$$

Sada je iz trokuta EFC:

$$\sin 18^\circ = \frac{|EF|}{|EC|} = \frac{\frac{3-\sqrt{5}}{4}}{\frac{-1+\sqrt{5}}{2}}$$

$$\sin 18^\circ = \frac{2(3-\sqrt{5})}{4(-1+\sqrt{5})} \cdot \frac{-1-\sqrt{5}}{-1-\sqrt{5}} = \frac{-3+\sqrt{5}-3\sqrt{5}+5}{2(1-5)} = \frac{2-2\sqrt{5}}{2\cdot(-4)} = \frac{2(\sqrt{5}-1)}{2\cdot 4}$$

$$\Longrightarrow \boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}.}$$

$$\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\frac{16-(5-2\sqrt{5}+1)}{16}}$$

$$\Longrightarrow \boxed{\cos 18^\circ = \frac{1}{4}\sqrt{10+2\sqrt{5}}.}$$

Sada koristimo

$$\sin\alpha = \sqrt{\frac{1-\cos 2\alpha}{2}}, \quad \cos\alpha = \sqrt{\frac{1+\cos 2\alpha}{2}}.$$

pa je

$$\sin 9^\circ = \sqrt{\frac{1-\cos 18^\circ}{2}} = \sqrt{\frac{1-\frac{1}{4}\sqrt{10+2\sqrt{5}}}{2}} = \sqrt{\frac{4-\sqrt{10+2\sqrt{5}}}{8}}$$

$$\Longrightarrow \boxed{\sin 9^\circ = \frac{1}{4}\sqrt{8-2\sqrt{10+2\sqrt{5}}}}$$

$$\cos 9^\circ = \sqrt{\frac{1+\cos 18^\circ}{2}} = \sqrt{\frac{1+\frac{1}{4}\sqrt{10+2\sqrt{5}}}{2}} = \sqrt{\frac{4+\sqrt{10+2\sqrt{5}}}{8}}$$

$$\Longrightarrow \boxed{\cos 9^\circ = \frac{1}{4}\sqrt{8+2\sqrt{10+2\sqrt{5}}}}$$

2. način.

Ako primijenimo formule:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

na jednakost

$$\sin 36^\circ = \cos 54^\circ$$

dobijemo

$$\begin{aligned} 2\sin 18^\circ \cos 18^\circ &= 4\cos^3 18^\circ - 3\cos 18^\circ \\ &\qquad \text{(dijelimo s } \cos 18^\circ \neq 0) \\ 2\sin 18^\circ &= 4\cos^2 18^\circ - 3 \\ 2\sin 18^\circ &= 4(1-\sin^2 18^\circ) - 3 \\ 2\sin 18^\circ &= 4 - 4\sin^2 18^\circ - 3 \\ 4\sin^2 18^\circ + 2\sin 18^\circ - 1 &= 0 \\ \sin 18^\circ &= \frac{-2+\sqrt{4-4\cdot 4\cdot(-1)}}{8} \\ &= \frac{-2+2\sqrt{5}}{8} \end{aligned}$$

$$\Longrightarrow \boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}}$$

(dalje analogno prethodnom načinu).

LITERATURA

1/ B. Dakić, N. Elezović: *Matematika 3 (1. dio)*, Element 2006.

2/ S. Mintaković, M. Franić: *Trigonometrija*, Element 1999.